Chapter 3

CAPM, Arbitrage, and Linear Factor Models

In this chapter, we analyze the asset pricing implications of the previous chapter's mean-variance portfolio analysis. From one perspective, the Markowitz-Tobin portfolio selection rules form a normative theory instructing how an individual investor can best allocate wealth among various assets. However, these selection rules also could be interpreted as a positive or descriptive theory of how an investor actually behaves. If this latter view is taken, then a logical extension of portfolio selection theory is to consider the equilibrium asset pricing consequences of investors' individually rational actions. The portfolio choices of individual investors represent their particular demands for assets. By aggregating these investor demands and equating them to asset supplies, equilibrium asset prices can be determined. In this way, portfolio choice theory can provide a foundation for an asset pricing model. Indeed, such a model, the Capital Asset Pricing Model (CAPM), was derived at about the same time by

four individuals: Jack Treynor, William Sharpe, John Lintner, and Jan Mossin.[1] CAPM has influenced financial practice in highly diverse ways. It has provided foundations for capital budgeting rules, for the regulation of utilities' rates of return, for performance evaluation of money managers, and for the creation of indexed mutual funds.

This chapter starts by deriving the CAPM and studying its consequences for assets' rates of return. The notion that investors might require higher rates of return for some types of risks but not others is an important insight of CAPM and extends to other asset pricing models. CAPM predicts that assets' risk premia result from a single risk factor, the returns on the market portfolio of all risky assets which, in equilibrium, is a mean-variance efficient portfolio. However, it is not hard to imagine that a weakening of CAPM's restrictive assumptions could generate risk premia deriving from multiple factors. Hence, we then consider how assets' risk premia may be related when multiple risk factors generate assets' returns. We derive this relationship, not based on a model of investor preferences as was done in deriving CAPM, but based on the concept that competitive and efficient securities markets should not permit arbitrage.

As a prelude to considering a multifactor asset pricing model, we define and give examples of arbitrage. Arbitrage pricing is the primary technique for valuing one asset in terms of another. It is the basis of so-called relative pricing models, contingent claims models, or derivative pricing models. We look at some simple applications of arbitrage pricing and then study the multifactor Arbitrage Pricing Theory (APT) developed by Stephen Ross (Ross 1976). APT is the basis of the most popular empirical multifactor models of asset pricing.

[1] William Sharpe, a student of Harry Markowitz, shared the 1990 Nobel prize with Markowitz and Merton Miller. See (Treynor 1961), (Sharpe 1964), (Lintner 1965), and (Mossin 1966).

3.1 The Capital Asset Pricing Model

In Chapter 2, we proved that if investors maximize expected utility that depends only on the expected return and variance of end-of-period wealth, then no matter what their particular levels of risk aversion, they would be interested only in portfolios on the efficient frontier. This mean-variance efficient frontier was the solution to the problem of computing portfolio weights that would maximize a portfolio's expected return for a given portfolio standard deviation or, alternatively, minimizing a portfolio's standard deviation for a given expected portfolio return. The point on this efficient frontier ultimately selected by a given investor was that combination of expected portfolio return and portfolio standard deviation that maximized the particular investor's expected utility. For the case of n risky assets and a risk-free asset, the optimal portfolio weights for the n risky assets were shown to be

$$\omega^* = \lambda V^{-1}\left(\bar{R} - R_f e\right) \tag{3.1}$$

where $\lambda \equiv \dfrac{\overline{R}_p - R_f}{\varsigma - 2\alpha R_f + \delta R_f^2}$, $\alpha \equiv \bar{R}' V^{-1} e = e' V^{-1} \bar{R}$, $\varsigma \equiv \bar{R}' V^{-1} \bar{R}$, and $\delta \equiv e' V^{-1} e$. The amount invested in the risk-free asset is then $1 - e'\omega^*$. Since λ is a scalar quantity that is linear in $\overline{R}_p$, which is the individual investor's equilibrium portfolio expected return, the weights in equation (3.1) are also linear in $\overline{R}_p$. $\overline{R}_p$ is determined by where the particular investor's indifference curve is tangent to the efficient frontier. Thus, because differences in $\overline{R}_p$ just affect the scalar, λ, we see that all investors, no matter what their degree of risk aversion, choose to hold the risky assets in the same *relative* proportions.

Mathematically, we showed that the efficient frontier is given by

$$\sigma_p = \frac{\overline{R}_p - R_f}{\left(\varsigma - 2\alpha R_f + \delta R_f^2\right)^{\frac{1}{2}}} \tag{3.2}$$

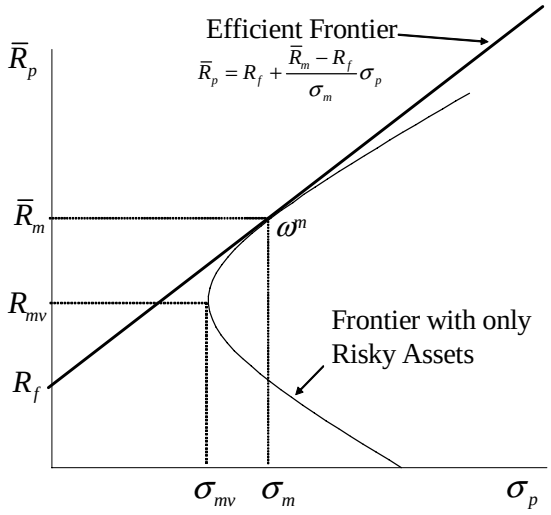


Figure 3.1: Capital Market Equilibrium

which, as illustrated in Figure 3.1, is linear when plotted in σ_p, $\overline{R}_p$ space.

3.1.1 Characteristics of the Tangency Portfolio

The efficient frontier, given by the line through R_f and ω^m, implies that investors optimally choose to hold combinations of the risk-free asset and the efficient frontier portfolio of risky assets having portfolio weights w^m. We can easily solve for this unique "tangency" portfolio of risky assets since it is the point where an investor would have a zero position in the risk-free asset; that is, $e'\omega^* = 1$, or $\overline{R}_p = \bar{R}'\omega^*$. Pre-multiplying (3.1) by e', setting the result to 1, and solving for λ, we obtain $\lambda = m \equiv [\alpha - \delta R_f]^{-1}$, so that

$$\omega^m = mV^{-1}(\bar{R} - R_f e) \tag{3.3}$$

Let us now investigate the relationship between this tangency portfolio and individual assets. Consider the covariance between the tangency portfolio and the individual risky assets. Define σ_M as the $n \times 1$ vector of covariances of the tangency portfolio with each of the n risky assets. Then using (3.3) we see that

$$\sigma_M = Vw^m = m(\bar{R} - R_f e) \tag{3.4}$$

Note that the variance of the tangency portfolio is simply $\sigma_m^2 = \omega^{m\prime} V \omega^m$. Accordingly, if we then pre-multiply equation (3.4) by $\omega^{m\prime}$, we obtain

$$\begin{aligned} \sigma_m^2 &= \omega^{m\prime}\sigma_M = m\omega^{m\prime}(\bar{R} - R_f e) \\ &= m(\bar{R}_m - R_f) \end{aligned} \tag{3.5}$$

where $\bar{R}_m \equiv \omega^{m\prime}\bar{R}$ is the expected return on the tangency portfolio.[2] Rearranging (3.4) and substituting in for m from (3.5), we have

$$(\bar{R} - R_f e) = \frac{1}{m}\sigma_M = \frac{\sigma_M}{\sigma_m^2}(\bar{R}_m - R_f) = \beta(\bar{R}_m - R_f) \tag{3.6}$$

where $\beta \equiv \frac{\sigma_M}{\sigma_m^2}$ is the $n \times 1$ vector whose i^{th} element is $\frac{Cov(\tilde{R}_m, \tilde{R}_i)}{Var(\tilde{R}_m)}$. Equation (3.6) shows that a simple relationship links the excess expected return (expected return in excess of the risk-free rate) on the tangency portfolio, $(\bar{R}_m - R_f)$, to the excess expected returns on the individual risky assets, $(\bar{R} - R_f e)$.

[2] Note that the elements of ω^m sum to 1 since the tangency portfolio has zero weight in the risk-free asset.

3.1.2 Market Equilibrium

Now suppose that individual investors, each taking the set of individual assets' expected returns and covariances as fixed (exogenous), all choose mean-variance efficient portfolios. Thus, each investor decides to allocate his or her wealth between the risk-free asset and the unique tangency portfolio. Because individual investors demand the risky assets in the same relative proportions, we know that the *aggregate demands* for the risky assets will have the same relative proportions, namely, those of the tangency portfolio. Recall that our derivation of this result *does not* assume a "representative" investor in the sense of requiring all investors to have identical utility functions or beginning-of-period wealth. It *does* assume that investors have identical beliefs regarding the probability distribution of asset returns, that all risky assets can be traded, that there are no indivisibilities in asset holdings, and that there are no limits on borrowing or lending at the risk-free rate.

We can now define an equilibrium as a situation where asset returns are such that the investors' demands for the assets equal the assets' supplies. What determines the assets' supplies? One way to model asset supplies is to assume they are fixed. For example, the economy could be characterized by a fixed *quantity* of physical assets that produce random output at the end of the period. Such an economy is often referred to as an *endowment economy*, and we detail a model of this type in Chapter 6. In this case, equilibrium occurs by adjustment of the date 0 assets' prices so that investors' demands conform to the inelastic assets' supplies. The change in the assets' date 0 prices effectively adjusts the assets' return distributions to those which make the tangency portfolio and the net demand for the risk-free asset equal to the fixed supplies of these assets.

An alternative way to model asset supplies is to assume that the economy's asset return distributions are fixed but allow the quantities of these assets to be

elastically supplied. This type of economy is known as a *production economy*, and a model of it is presented in Chapter 13. Such a model assumes that there are n risky, constant-returns-to-scale "technologies." These technologies require date 0 investments of physical capital and produce end-of-period physical investment returns having a distribution with mean $\bar{R}$ and a covariance matrix of V at the end of the period. Also, there could be a risk-free technology that generates a one-period return on physical capital of R_f. In this case of a fixed return distribution, supplies of the assets adjust to the demands for the tangency portfolio and the risk-free asset determined by the technological return distribution.

As it turns out, how one models asset supplies does not affect the results that we now derive regarding the equilibrium relationship between asset returns. We simply note that the tangency portfolio having weights ω^m must be the equilibrium portfolio of risky assets supplied in the market. Thus, equation (3.6) can be interpreted as an equilibrium relationship between the excess expected return on any asset and the excess expected return on the *market* portfolio. In other words, in equilibrium, the tangency portfolio chosen by all investors must be the market portfolio of all risky assets. Moreover, as mentioned earlier, the only case for which investors have a long position in the tangency portfolio is $R_f < R_{mv}$. Hence, for asset markets to clear, that is, for the outstanding stocks of assets to be owned by investors, the situation depicted in Figure 3.1 can be the only equilibrium efficient frontier.[3]

The Capital Asset Pricing Model's prediction that the market portfolio is mean-variance efficient is an important solution to the practical problem of identifying a mean-variance efficient portfolio. As a theory, CAPM justifies the

[3] This presumes that the tangency portfolio is composed of long positions in the individual risky assets; that is, $\omega_i^m > 0$ for $i = 1, ..., n$. While our derivation has not restricted the sign of these portfolio weights, since assets must have nonnegative supplies, equilibrium market clearing implies that assets' prices or individuals' choice of technologies must adjust (effectively changing $\overline{R}$ and/or V) to make the portfolio demands for individual assets nonnegative.

practice of investing in a broad market portfolio of stocks and bonds. This insight has led to the growth of "indexed" mutual funds and exchange-traded funds (ETFs) that hold market-weighted portfolios of stocks and bonds.

Let's now look at some additional implications of CAPM when we consider realized, rather than expected, asset returns. Note that asset i's realized return, $\tilde{R}_i$, can be defined as $\bar{R}_i + \tilde{\nu}_i$, where $\tilde{\nu}_i$ is the unexpected component of the asset's return. Similarly, the realized return on the market portfolio, $\tilde{R}_m$, can be defined as $\bar{R}_m + \tilde{\nu}_m$, where $\tilde{\nu}_m$ is the unexpected part of the market portfolio's return. Substituting these into (3.6), we have

$$\begin{aligned} \tilde{R}_i &= R_f + \beta_i(\tilde{R}_m - \tilde{\nu}_m - R_f) + \tilde{\nu}_i \\ &= R_f + \beta_i(\tilde{R}_m - R_f) + \tilde{\nu}_i - \beta_i\tilde{\nu}_m \\ &= R_f + \beta_i(\tilde{R}_m - R_f) + \widetilde{\varepsilon}_i \end{aligned} \tag{3.7}$$

where $\widetilde{\varepsilon}_i \equiv \tilde{\nu}_i - \beta_i\tilde{\nu}_m$. Note that

$$\begin{aligned} Cov(\tilde{R}_m, \widetilde{\varepsilon}_i) &= Cov(\tilde{R}_m, \tilde{\nu}_i) - \beta_i Cov(\tilde{R}_m, \tilde{\nu}_m) \\ &= Cov(\tilde{R}_m, \tilde{R}_i) - \beta_i Cov(\tilde{R}_m, \tilde{R}_m) \\ &= \beta_i Var(\tilde{R}_m) - \beta_i Var(\tilde{R}_m) = 0 \end{aligned} \tag{3.8}$$

which, along with (3.7), implies that the total variance of risky asset i, σ_i^2, has two components:

$$\sigma_i^2 = \beta_i^2\sigma_m^2 + \sigma_{\varepsilon_i}^2 \tag{3.9}$$

where $\beta_i^2\sigma_m^2$ is proportional to the return variance of the market portfolios and $\sigma_{\varepsilon_i}^2$ is the variance of $\widetilde{\varepsilon}_i$, and it is orthogonal to the market portfolio's return.

Since equation (3.8) shows that $\widetilde{\varepsilon}_i$ is the part of the return on risky asset i that is uncorrelated with the return on the market portfolio, this implies that equation (3.7) represents a regression equation. In other words, an unbiased estimate of β_i can be obtained by running an Ordinary Least Squares regression of asset i's excess return on the market portfolio's excess return. The orthogonal, mean-zero residual, $\widetilde{\varepsilon}_i$, is sometimes referred to as idiosyncratic, unsystematic, or diversifiable risk. This is the particular asset's risk that is eliminated or diversified away when the asset is held in the market portfolio. Since this portion of the asset's risk can be eliminated by the individual who invests optimally, there is no "price" or "risk premium" attached to it in the sense that the asset's equilibrium expected return is not altered by it.

To make clear what risk is priced, let us denote the covariance between the return on the i^{th} asset and the return on the market portfolio as $\sigma_{Mi} = Cov(\tilde{R}_m, \widetilde{R}_i)$, which is the i^{th} element of σ_M. Also let ρ_{im} be the correlation between the return on the i^{th} asset and the return on the market portfolio. Then equation (3.6) can be rewritten as

$$\begin{aligned} \bar{R}_i - R_f &= \frac{\sigma_{Mi}}{\sigma_m}\frac{(\bar{R}_m - R_f)}{\sigma_m} \\ &= \rho_{im}\sigma_i\frac{(\bar{R}_m - R_f)}{\sigma_m} \\ &= \rho_{im}\sigma_i S_e \end{aligned} \tag{3.10}$$

where $S_e \equiv \frac{(\bar{R}_m - R_f)}{\sigma_m}$ is the equilibrium excess return on the market portfolio per unit of market risk and is known as the market Sharpe ratio, named after William Sharpe, one of the developers of the CAPM. S_e can be interpreted as the market price of systematic or nondiversifiable risk. It is also referred to as the slope of the *capital market line*, where the capital market line is defined as

the efficient frontier that connects the points R_f and ω^m in Figure 3.1. Now if we define ω_i^m as the weight of asset i in the market portfolio and V_i as the i^{th} row of covariance matrix V, then

$$\frac{\partial \sigma_m}{\partial \omega_i^m} = \frac{1}{2\sigma_m}\frac{\partial \sigma_m^2}{\partial \omega_i^m} = \frac{1}{2\sigma_m}\frac{\partial \omega^m V \omega^m}{\partial \omega_i^m} = \frac{1}{2\sigma_m} 2V_i\omega^m = \frac{1}{\sigma_m}\sum_{j=1}^{n}\omega_j^m\sigma_{ij} \qquad (3.11)$$

where σ_{ij} is the i,j^{th} element of V. Since $\tilde{R}_m = \sum_{j=1}^{n}\omega_j^m\tilde{R}_j$, then $Cov(\tilde{R}_i, \tilde{R}_m) = Cov(\tilde{R}_i, \sum_{j=1}^{n}\omega_j^m\tilde{R}_j) = \sum_{j=1}^{n}\omega_j^m\sigma_{ij}$. Hence, (3.11) can be rewritten as

$$\frac{\partial \sigma_m}{\partial \omega_i^m} = \frac{1}{\sigma_m}Cov(\tilde{R}_i, \tilde{R}_m) = \rho_{im}\sigma_i \qquad (3.12)$$

Thus, $\rho_{im}\sigma_i$ can be interpreted as the marginal increase in "market risk," σ_m, from a marginal increase of asset i in the market portfolio. In this sense, $\rho_{im}\sigma_i$ is the *quantity* of asset i's systematic or nondiversifiable risk. Equation (3.10) shows that this quantity of systematic risk, multiplied by the price of systematic risk, S_e, determines the asset's required excess expected return, or risk premium.

If a riskless asset does not exist so that all assets are risky, Fischer Black (Black 1972) showed that a similar asset pricing relationship exists. Here, we outline his *zero-beta CAPM*. Note that an implication of the portfolio separation result of section 2.3.3 is that since every frontier portfolio can be written as $\omega = a + b\bar{R}_p$, a linear combination of these frontier portfolios is also a frontier portfolio. Let W_i be the proportion of the economy's total wealth owned by investor i, and let ω^i be this investor's desired frontier portfolio so that $\omega^i = a + b\bar{R}_{ip}$. If there are a total of I investors, then the weights of the market portfolio are given by

$$\begin{aligned} \omega^m &= \sum_{i=1}^{I} W_i \omega^i = \sum_{i=1}^{I} W_i \left(a + b\bar{R}_{ip}\right) \\ &= a\sum_{i=1}^{I} W_i + b\sum_{i=1}^{I} W_i \bar{R}_{ip} = a + b\bar{R}_m \end{aligned} \tag{3.13}$$

where $\bar{R}_m \equiv \sum_{i=1}^{I} W_i \bar{R}_{ip}$ and where the last equality of (3.13) uses the fact that the sum of the proportions of total wealth must equal 1. Equation (3.13) shows that the market portfolio, the aggregation of all individual investors' portfolios, is a frontier portfolio. Its expected return, $\overline{R}_m$, is a weighted average of the expected returns of the individual investors' portfolios. Because each individual investor optimally chooses a portfolio on the efficient portion of the frontier (the upper arc in Figure 2.4), then the market portfolio, being a weighted average, is also on the efficient frontier.

Now, let us compute the covariance between the market portfolio and any arbitrary portfolio of risky assets, not necessarily a frontier portfolio. Let this arbitrary risky-asset portfolio have weights ω^0, a random return of $\widetilde{R}_{0p}$, and an expected return of $\overline{R}_{0p}$. Then

$$\begin{aligned} Cov\left(\widetilde{R}_m, \widetilde{R}_{0p}\right) &= \omega^{m\prime} V \omega^0 = \left(a + b\bar{R}_m\right)' V\omega^0 \\ &= \left(\frac{\varsigma V^{-1}e - \alpha V^{-1}\bar{R}}{\varsigma\delta - \alpha^2} + \frac{\delta V^{-1}\bar{R} - \alpha V^{-1}e}{\varsigma\delta - \alpha^2}\bar{R}_m\right)' V\omega^0 \\ &= \frac{\varsigma e' V^{-1} V\omega^0 - \alpha\bar{R}' V^{-1} V\omega^0}{\varsigma\delta - \alpha^2} \\ &\quad + \frac{\delta\bar{R}_m\bar{R}' V^{-1} V\omega^0 - \alpha\bar{R}_m e' V^{-1} V\omega^0}{\varsigma\delta - \alpha^2} \\ &= \frac{\varsigma - \alpha\overline{R}_{0p} + \delta\bar{R}_m\overline{R}_{0p} - \alpha\bar{R}_m}{\varsigma\delta - \alpha^2} \end{aligned} \tag{3.14}$$

Rearranging (3.14) gives

$$\overline{R}_{0p} = \frac{\alpha \bar{R}_m - \varsigma}{\delta \bar{R}_m - \alpha} + Cov\left(\widetilde{R}_m, \widetilde{R}_{0p}\right) \frac{\varsigma\delta - \alpha^2}{\delta \bar{R}_m - \alpha} \tag{3.15}$$

Rewriting the first term on the right-hand side of equation (3.15) and multiplying and dividing the second term by the definition of a frontier portfolio's variance given in Chapter 2's equation (2.32), equation (3.15) becomes

$$\begin{aligned} \overline{R}_{0p} &= \frac{\alpha}{\delta} - \frac{\varsigma\delta - \alpha^2}{\delta^2\left(\overline{R}_m - \frac{\alpha}{\delta}\right)} + \frac{Cov\left(\widetilde{R}_m, \widetilde{R}_{0p}\right)}{\sigma_m^2}\left(\frac{1}{\delta} + \frac{\delta\left(\overline{R}_m - \frac{\alpha}{\delta}\right)^2}{\varsigma\delta - \alpha^2}\right)\frac{\varsigma\delta - \alpha^2}{\delta\bar{R}_m - \alpha} \\ &= \frac{\alpha}{\delta} - \frac{\varsigma\delta - \alpha^2}{\delta^2\left(\overline{R}_m - \frac{\alpha}{\delta}\right)} + \frac{Cov\left(\widetilde{R}_m, \widetilde{R}_{0p}\right)}{\sigma_m^2}\left(\overline{R}_m - \frac{\alpha}{\delta} + \frac{\varsigma\delta - \alpha^2}{\delta^2\left(\overline{R}_m - \frac{\alpha}{\delta}\right)}\right) \end{aligned} \tag{3.16}$$

From equation (2.39), we recognize that the first two terms on the right-hand side of (3.16) equal the expected return on the portfolio that has zero covariance with the market portfolio, call it $\overline{R}_{zm}$. Thus, equation (3.16) can be written as

$$\begin{aligned} \overline{R}_{0p} &= \overline{R}_{zm} + \frac{Cov\left(\widetilde{R}_m, \widetilde{R}_{0p}\right)}{\sigma_m^2}\left(\overline{R}_m - \overline{R}_{zm}\right) \\ &= \overline{R}_{zm} + \beta_0\left(\overline{R}_m - \overline{R}_{zm}\right) \end{aligned} \tag{3.17}$$

Since the portfolio having weights ω^0 can be any risky-asset portfolio, it includes a portfolio that invests solely in a single asset.[4] In this light, β_0 becomes the covariance of the individual asset's return with that of the market portfolio, and the relationship in equation (3.17) is identical to the previous CAPM result in equation (3.10) except that $\overline{R}_{zm}$ replaces R_f. Hence, when a riskless asset does not exist, we measure an asset's excess returns relative to $\overline{R}_{zm}$, the expected return on a portfolio that has a zero beta.

[4] One of the elements of ω^0 would equal 1, while the rest would be zero.

Because the CAPM relationship in equations (3.10) or (3.17) implies that assets' expected returns differ only due to differences in their betas, it is considered a single "factor" model, this risk factor being the return on the market portfolio. Stephen Ross (Ross 1976) derived a similar multifactor relationship, but starting from a different set of assumptions and using a derivation based on the arbitrage principle. Frequently in this book, we will see that asset pricing implications can often be derived based on investor risk preferences, as was done in the CAPM when we assumed investors cared only about the mean and variance of their portfolio's return. However, another powerful technique for asset pricing is to rule out the existence of arbitrage. We now turn to this topic, first by discussing the nature of arbitrage.

3.2 Arbitrage

The notion of arbitrage is simple. It involves the possibility of getting something for nothing while having no possibility of loss. Specifically, consider constructing a portfolio involving both long and short positions in assets such that no initial wealth is required to form the portfolio.[5] If this zero-net-investment portfolio can sometimes produce a positive return but can never produce a negative return, then it represents an arbitrage: starting from zero wealth, a profit can sometimes be made but a loss can never occur. A special case of arbitrage is when this zero-net-investment portfolio produces a riskless return. If this certain return is positive (*negative*), an arbitrage is to buy (*sell*) the portfolio and reap a riskless profit, or "free lunch." Only if the return is zero would there be no arbitrage.

An arbitrage opportunity can also be defined in a slightly different context.

[5] Proceeds from short sales (or borrowing) are used to purchase (take long positions in) other assets.

If a portfolio that requires a nonzero initial net investment is created such that it earns a certain rate of return, then this rate of return must equal the current (competitive market) risk-free interest rate. Otherwise, there would also be an arbitrage opportunity. For example, if the portfolio required a positive initial investment but earned less than the risk-free rate, an arbitrage would be to (short-) sell the portfolio and invest the proceeds at the risk-free rate, thereby earning a riskless profit equal to the difference between the risk-free rate and the portfolio's certain (lower) rate of return.[6]

In efficient, competitive asset markets where arbitrage trades are feasible, it is reasonable to think that arbitrage opportunities are rare and fleeting. Should arbitrage temporarily exist, then trading by investors to earn this profit will tend to move asset prices in a direction that eliminates the arbitrage opportunity. For example, if a zero-net-investment portfolio produces a riskless positive return, as investors create (buy) this portfolio, the prices of the assets in the portfolio will be bid up. The cost of creating the portfolio will then exceed zero. The portfolio's cost will rise until it equals the present value of the portfolio's riskless return, thereby eliminating the arbitrage opportunity. Hence, for competitive asset markets where it is also feasible to execute arbitrage trades, it may be reasonable to assume that equilibrium asset prices reflect an absence of arbitrage opportunities. As will be shown, this assumption leads to a law of one price: if different assets produce exactly the same future payoffs, then the current prices of these assets must be the same. This simple result has powerful asset pricing implications.

[6] Arbitrage defined in this context is really equivalent to the previous definition of arbitrage. For example, if a portfolio requiring a positive initial investment produces a certain rate of return in excess of the riskless rate, then an investor should be able to borrow the initial funds needed to create this portfolio and pay an interest rate on this loan that equals the risk-free interest rate. That the investor should be able to borrow at the riskless interest rate can be seen from the fact that the portfolio produces a return that is always sufficient to repay the loan in full, making the borrowing risk-free. Hence, combining this initial borrowing with the nonzero portfolio investment results in an arbitrage opportunity that requires zero initial wealth.

However, as a word of caution, not all asset markets meet the conditions required to justify arbitrage pricing. For some markets, it may be impossible to execute pure arbitrage trades due to significant transactions costs and/or restrictions on short-selling or borrowing. In such cases of limited arbitrage, the law of one price can fail.[7] Alternative methods, such as those based on a model of investor preferences, are required to price assets.

3.2.1 Examples of Arbitrage Pricing

An early use of the arbitrage principle is the *covered interest parity* condition that links spot and forward foreign exchange markets to foreign and domestic money markets. To illustrate, let $F_{0\tau}$ be the current date 0 forward price for exchanging one unit of a foreign currency τ periods in the future. This forward price represents the dollar price to be paid τ periods in the future for delivery of one unit of foreign currency τ periods in the future. In contrast, let S_0 be the spot price of foreign exchange, that is, the current date 0 dollar price of one unit of foreign currency to be delivered immediately. Also let R_f be the per-period risk-free (money market) return for borrowing or lending in dollars over the period 0 to τ, and denote as R_f^* the per-period risk-free return for borrowing or lending in the foreign currency over the period 0 to τ.[8]

Now construct the following portfolio that requires zero net wealth. First, we sell forward (take a short forward position in) one unit of foreign exchange at price $F_{0\tau}$.[9] This contract means that we are committed to delivering one unit of foreign exchange at date τ in return for receiving $F_{0\tau}$ dollars at date τ. Second, let us also purchase the present value of one unit of foreign currency,

[7] Andrei Shleifer and Robert Vishny (Shleifer and Vishny 1997) discuss why the conditions needed to apply arbitrage pricing are not present in many asset markets.

[8] For example, if the foreign currency is the Japanese yen, R_f^* would be the per-period return for a yen-denominated risk-free investment or loan.

[9] Taking a long or short position in a forward contract requires zero initial wealth, as payment and delivery all occur at the future date τ.

$1/R_f^{*\tau}$, and invest it in a foreign bond yielding the per-period return, R_f^*. In terms of the domestic currency, this purchase costs $S_0/R_f^{*\tau}$, which we finance by borrowing dollars at the per-period return R_f.

What happens at date τ as a result of these trades? When date τ arrives, we know that our foreign currency investment yields $R_f^{*\tau}/R_f^{*\tau} = 1$ unit of the foreign currency. This is exactly what we need to satisfy our short position in the forward foreign exchange contract. For delivering this foreign currency, we receive $F_{0\tau}$ dollars. But we also now owe a sum of $R_f^\tau S_0/R_f^{*\tau}$ due to our dollar borrowing. Thus, our net proceeds at date τ are

$$F_{0\tau} - R_f^\tau S_0/R_f^{*\tau} \tag{3.18}$$

Note that these proceeds are nonrandom; that is, the amount is known at date 0 since it depends only on prices and riskless rates quoted at date 0. If this amount is positive, then we should indeed create this portfolio as it represents an arbitrage. If, instead, this amount is negative, then an arbitrage would be for us to sell this portfolio; that is, we reverse each trade just discussed (i.e., take a long forward position, and invest in the domestic currency financed by borrowing in foreign currency markets). Thus, the only instance in which arbitrage would not occur is if the net proceeds are zero, which implies

$$F_{0\tau} = S_0 R_f^\tau / R_f^{*\tau} \tag{3.19}$$

Equation (3.19) is referred to as the *covered interest parity* condition.

The forward exchange rate, $F_{0\tau}$, represents the dollar price for buying or selling a foreign currency at date τ, a future date when the foreign currency's dollar value is unknown. Though $F_{0\tau}$ is the price of a risky cashflow, it has been determined without knowledge of the utility functions of investors or their

expectations regarding the future value of the foreign currency. The reason for this simplification is due to the *law of one price*, which states that in the absence of arbitrage, equivalent assets (or contracts) must have the same price. A forward contract to purchase a unit of foreign currency can be replicated by buying, at the spot exchange rate S_0, a foreign currency investment paying the per-period, risk-free return R_f^* and financing this by borrowing at the dollar risk-free return R_f. In the absence of arbitrage, these two methods for obtaining foreign currency in the future must be valued the same. Given the spot exchange rate, S_0, and the foreign and domestic money market returns, R_f^* and R_f, the forward rate is pinned down. Thus, when applicable, pricing assets or contracts by ruling out arbitrage is attractive in that assumptions regarding investor preferences or beliefs are not required.

To motivate how arbitrage pricing might apply to a very simple version of the CAPM, suppose that there is a risk-free asset that returns R_f and multiple risky assets. However, assume that only a single source of (market) risk determines all risky-asset returns and that these returns can be expressed by the linear relationship

$$\widetilde{R}_i = a_i + b_i \tilde{f} \tag{3.20}$$

where $\widetilde{R}_i$ is the return on the i^{th} asset and $\tilde{f}$ is the single risk factor generating all asset returns, where it is assumed that $E[\tilde{f}] = 0$. a_i is asset i's expected return, that is, $E[\widetilde{R}_i] = a_i$. b_i is the sensitivity of asset i to the risk factor and can be viewed as asset i's beta coefficient. Note that this is a highly simplified example in that all risky assets are perfectly correlated with each other. Assets have no idiosyncratic risk (residual component $\widetilde{\varepsilon}_i$). A generalized model with idiosyncratic risk will be presented in the next section.

Now suppose that a portfolio of two assets is constructed, where a proportion

of wealth of ω is invested in asset i and the remaining proportion of $(1-\omega)$ is invested in asset j. This portfolio's return is given by

$$\begin{aligned} \widetilde{R}_p &= \omega a_i + (1-\omega)a_j + \omega b_i \tilde{f} + (1-\omega) b_j \tilde{f} \\ &= \omega(a_i - a_j) + a_j + [\omega(b_i - b_j) + b_j]\, \tilde{f} \end{aligned} \tag{3.21}$$

If the portfolio weights are chosen such that

$$\omega^* = \frac{b_j}{b_j - b_i} \tag{3.22}$$

then the uncertain (random) component of the portfolio's return is eliminated. The absence of arbitrage then requires that $R_p = R_f$, so that

$$R_p = \omega^*(a_i - a_j) + a_j = R_f \tag{3.23}$$

or

$$\frac{b_j(a_i - a_j)}{b_j - b_i} + a_j = R_f$$

which implies

$$\frac{a_i - R_f}{b_i} = \frac{a_j - R_f}{b_j} \equiv \lambda \tag{3.24}$$

This condition states that the expected return in excess of the risk-free rate, per unit of risk, must be equal for all assets, and we define this ratio as λ. λ is the risk premium per unit of the factor risk. The denominator, b_i, can be interpreted as asset i's quantity of risk from the single risk factor, while $a_i - R_f$ can be thought of as asset i's compensation or premium in terms of excess expected return given to investors for holding asset i. Thus, this no-arbitrage

condition is really a law of one price in that the *price of risk*, λ, which is the risk premium divided by the quantity of risk, must be the same for all assets.

Equation (3.24) is a fundamental relationship, and similar law-of-one-price conditions hold for virtually all asset pricing models. For example, we can rewrite the CAPM equation (3.10) as

$$\frac{\bar{R}_i - R_f}{\rho_{im}\sigma_i} = \frac{(\bar{R}_m - R_f)}{\sigma_m} \equiv S_e \tag{3.25}$$

so that the ratio of an asset's expected return premium, $\bar{R}_i - R_f$, to its quantity of market risk, $\rho_{im}\sigma_i$, is the same for all assets and equals the slope of the capital market line, S_e. We next turn to a generalization of the CAPM that derives from arbitrage pricing.

3.3 Linear Factor Models

The CAPM assumption that all assets can be held by all individual investors is clearly an oversimplification. Transactions costs and other trading "frictions" that arise from distortions such as capital controls and taxes might prevent individuals from holding a global portfolio of marketable assets. Furthermore, many assets simply are nonmarketable and cannot be traded.[10] The preeminent example of a nonmarketable asset is the value of an individual's future labor income, what economists refer to as the individual's *human capital*. Therefore, in addition to the risk from returns on a global portfolio of marketable assets, individuals are likely to face multiple sources of nondiversifiable risks. It is then not hard to imagine that, in equilibrium, assets' risk premia derive from

[10] Richard Roll (Roll 1977) has argued that CAPM is not a reasonable theory, because a true "market" portfolio consisting of all risky assets cannot be observed or owned by investors. Moreover, empirical tests of CAPM are infeasible because proxies for the market portfolio (such as the S&P 500 stock index) may not be mean-variance efficient, even if the true market portfolio is. Conversely, a proxy for the market portfolio could be mean-variance efficient even though the true market portfolio is not.

more than a single risk factor. Indeed, the CAPM's prediction that risk from a market portfolio is the only source of priced risk has not received strong empirical support.[11]

This is a motivation for the multifactor Arbitrage Pricing Theory (APT) model. APT assumes that an individual asset's return is driven by multiple risk factors and by an idiosyncratic component, though the theory is mute regarding the sources of these multiple risk factors. APT is a relative pricing model in the sense that it determines the risk premia on all assets relative to the risk premium for each of the factors and each asset's sensitivity to each factor.[12] It does not make assumptions regarding investor preferences but uses arbitrage pricing to restrict an asset's risk premium. The main assumptions of the model are that the returns on all assets are linearly related to a finite number of risk factors and that the number of assets in the economy is large relative to the number of factors. Let us now detail the model's assumptions.

Assume that there are k risk factors and n assets in the economy, where $n > k$. Let b_{iz} be the sensitivity of the i^{th} asset to the z^{th} risk factor, where $\tilde{f}_z$ is the random realization of risk factor z. Also let $\widetilde{\varepsilon}_i$ be the idiosyncratic risk component specific to asset i, which by definition is independent of the k risk factors, $\tilde{f}_1$,...,$\tilde{f}_k$, and the specific risk component of any other asset j, $\widetilde{\varepsilon}_j$. $\widetilde{\varepsilon}_i$ must be independent of the risk factors or else it would affect all assets, thus not being truly a specific source of risk to just asset i. If a_i is the expected return on asset i, then the return-generating process for asset i is given by the linear factor model

[11] Ravi Jagannathan and Ellen McGrattan (Jagannathan and McGrattan 1995) review the empirical evidence for CAPM.

[12] This is not much different from the CAPM. CAPM determined each asset's risk premium based on the single-factor market risk premium, $\overline{R}_m - R_f$, and the asset's sensitivity to this single factor, β_i. The only difference is that CAPM provides somewhat more guidance as to the identity of the risk factor, namely, the return on a market portfolio of all assets.

$$\widetilde{R}_i = a_i + \sum_{z=1}^{k} b_{iz}\tilde{f}_z + \widetilde{\varepsilon}_i \tag{3.26}$$

where $E[\widetilde{\varepsilon}_i] = E\left[\tilde{f}_z\right] = E\left[\widetilde{\varepsilon}_i\tilde{f}_z\right] = 0$, and $E[\widetilde{\varepsilon}_i\widetilde{\varepsilon}_j] = 0$ for $i \neq j$. For simplicity, we also assume that $E\left[\tilde{f}_z\tilde{f}_x\right] = 0$ for $z \neq x$; that is, the risk factors are mutually independent. In addition, let us further assume that the risk factors are normalized to have a variance equal to one, so that $E\left[\tilde{f}_z^2\right] = 1$. As it turns out, these last two assumptions are not important, as a linear transformation of correlated risk factors can allow them to be redefined as independent, unit-variance risk factors.[13]

A final assumption is that the idiosyncratic risk (variance) for each asset is finite; that is,

$$E\left[\widetilde{\varepsilon}_i^2\right] \equiv s_i^2 < S^2 \tag{3.27}$$

where S^2 is some finite number. Under these assumptions, note that $Cov\left(\widetilde{R}_i, \tilde{f}_z\right) = Cov\left(b_{iz}\tilde{f}_z, \tilde{f}_z\right) = b_{iz}Cov\left(\tilde{f}_z, \tilde{f}_z\right) = b_{iz}$. Thus, b_{iz} is the covariance between the return on asset i and factor z.

In the simple example of the previous section, assets had no idiosyncratic risk, and their expected returns could be determined by ruling out a simple arbitrage. This was because a hedge portfolio, consisting of appropriate combinations of different assets, could be created that had a riskless return. Now, however, when each asset's return contains an idiosyncratic risk component, it is not possible to create a hedge portfolio having a purely riskless return. Instead, we will argue that if the number of assets is large, a portfolio can be constructed that has "close" to a riskless return, because the idiosyncratic components of

[13] For example, suppose $\widetilde{g}$ is a $k \times 1$ vector of mean-zero, correlated risk factors with $k \times k$ covariance matrix $E[\widetilde{g}\widetilde{g}'] = \Omega$. Then create a transformed $k \times 1$ vector of risk factors given by $\widetilde{f} = \sqrt{\Omega^{-1}}\widetilde{g}$. The covariance matrix of these transformed risk factors is $E\left[\widetilde{f}\widetilde{f}'\right] = \sqrt{\Omega^{-1}}E[\widetilde{g}\widetilde{g}']\sqrt{\Omega^{-1}} = I_k$ where I_k is a $k \times k$ identity matrix.

assets' returns are diversifiable. While ruling out pure arbitrage opportunities is not sufficient to constrain assets' expected returns, we can use the notion of *asymptotic arbitrage* to argue that assets' expected returns will be "close" to the relationship that would result if they had no idiosyncratic risk. So let us now state what we mean by an asymptotic arbitrage opportunity.[14]

Definition: Let a portfolio containing n assets be described by the vector of investment amounts in each of the n assets, $W^n \equiv [W_1^n \; W_2^n \; ... \; W_n^n]'$. Thus, W_i^n is the amount invested in asset i when there are n total assets in the economy. Consider a sequence of these portfolios where n is increasing, $n = 2, 3, \ldots$. Let σ_{ij} be the covariance between the returns on assets i and j. Then an asymptotic arbitrage exists if the following conditions hold:

(A) The portfolio requires zero net investment:

$$\sum_{i=1}^{n} W_i^n = 0$$

(B) The portfolio return becomes certain as n gets large:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{n} W_i^n W_j^n \sigma_{ij} = 0$$

(C) The portfolio's expected return is always bounded above zero

$$\sum_{i=1}^{n} W_i^n a_i \geq \delta > 0$$

We can now state the Arbitrage Pricing Theorem (APT):

Theorem: If no asymptotic arbitrage opportunities exist, then the expected return of asset i, $i = 1, ..., n$, is described by the following linear relation:

[14] This proof of Arbitrage Pricing Theory based on the concept of asymptotic arbitrage is due to Gur Huberman (Huberman 1982).

$$a_i = \lambda_0 + \sum_{z=1}^{k} b_{iz}\lambda_z + \nu_i \qquad (*)$$

where λ_0 is a constant, λ_z is the risk premium for risk factor $\widetilde{f}_z$, $z = 1, ..., k$, and the expected return deviations, ν_i, satisfy

$$\sum_{i=1}^{n} \nu_i = 0 \qquad (i)$$

$$\sum_{i=1}^{n} b_{iz}\nu_i = 0, \quad z = 1, ..., k \qquad (ii)$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \nu_i^2 = 0 \qquad (iii)$$

Note that condition (iii) says that the average squared error (deviation) from the pricing rule $(*)$ goes to zero as n becomes large. Thus, as the number of assets increases relative to the risk factors, expected returns will, on average, become closely approximated by the relation $a_i = \lambda_0 + \sum_{z=1}^{k} b_{iz}\lambda_z$. Also note that if the economy contains a risk-free asset (implying $b_{iz} = 0, \forall\, z$), the risk-free return will be approximated by λ_0.

Proof: For a given number of assets, $n > k$, think of running a cross-sectional regression of the a_i's on the b_{iz}'s. More precisely, project the dependent variable vector $a = [a_1\ a_2 \dots a_n]'$ on the k explanatory variable vectors $b_z = [b_{1z}\ b_{2z} \dots b_{nz}]$, $z = 1, ..., k$. Define ν_i as the regression residual for observation i, $i = 1, ..., n$. Denote λ_0 as the regression intercept and λ_z, $z = 1, ..., k$, as the estimated coefficient on explanatory variable z. The regression estimates and residuals must then satisfy

$$a_i = \lambda_0 + \sum_{z=1}^{k} b_{iz}\lambda_z + \nu_i \qquad (3.28)$$

where by the properties of an orthogonal projection (Ordinary Least Squares regression), the residuals sum to zero, $\sum_{i=1}^{n} \nu_i = 0$, and are orthogonal to the regressors, $\sum_{i=1}^{n} b_{iz}\nu_i = 0$, $z = 1, ..., k$. Thus, we have shown that $(*)$, (i), and (ii) can be satisfied. The last but most important part of the proof is to show that (iii) must hold in the absence of asymptotic arbitrage.

Thus, let us construct a zero-net-investment arbitrage portfolio with the following investment amounts:

$$W_i = \frac{\nu_i}{\sqrt{\sum_{i=1}^{n} \nu_i^2 n}} \tag{3.29}$$

so that greater amounts are invested in assets having the greatest relative expected return deviation. The total arbitrage portfolio return is given by

$$\begin{aligned} \tilde{R}_p &= \sum_{i=1}^{n} W_i \widetilde{R}_i \qquad (3.30) \\ &= \frac{1}{\sqrt{\sum_{i=1}^{n} \nu_i^2 n}} \left[\sum_{i=1}^{n} \nu_i \widetilde{R}_i \right] = \frac{1}{\sqrt{\sum_{i=1}^{n} \nu_i^2 n}} \left[\sum_{i=1}^{n} \nu_i \left(a_i + \sum_{z=1}^{k} b_{iz} \tilde{f}_z + \widetilde{\varepsilon}_i \right) \right] \end{aligned}$$

Since $\sum_{i=1}^{n} b_{iz}\nu_i = 0$, $z = 1, ..., k$, this equals

$$\tilde{R}_p = \frac{1}{\sqrt{\sum_{i=1}^{n} \nu_i^2 n}} \left[\sum_{i=1}^{n} \nu_i \left(a_i + \widetilde{\varepsilon}_i \right) \right] \tag{3.31}$$

Let us calculate this portfolio's mean and variance. Taking expectations, we obtain

$$E\left[\tilde{R}_p\right] = \frac{1}{\sqrt{\sum_{i=1}^{n} \nu_i^2 n}} \left[\sum_{i=1}^{n} \nu_i a_i \right] \tag{3.32}$$

since $E[\widetilde{\varepsilon}_i] = 0$. Substituting in for $a_i = \lambda_0 + \sum_{z=1}^{k} b_{iz}\lambda_z + \nu_i$, we have

$$E\left[\tilde{R}_p\right] = \frac{1}{\sqrt{\sum_{i=1}^{n} \nu_i^2 n}} \left[\lambda_0 \sum_{i=1}^{n} \nu_i + \sum_{z=1}^{k} \left(\lambda_z \sum_{i=1}^{n} \nu_i b_{iz}\right) + \sum_{i=1}^{n} \nu_i^2\right] \tag{3.33}$$

and since $\sum_{i=1}^{n} \nu_i = 0$ and $\sum_{i=1}^{n} \nu_i b_{iz} = 0$, this simplifies to

$$E\left[\tilde{R}_p\right] = \frac{1}{\sqrt{\sum_{i=1}^{n} \nu_i^2 n}} \sum_{i=1}^{n} \nu_i^2 = \sqrt{\frac{1}{n} \sum_{i=1}^{n} \nu_i^2} \tag{3.34}$$

To calculate the portfolio's variance, start by subtracting (3.32) from (3.31):

$$\tilde{R}_p - E\left[\tilde{R}_p\right] = \frac{1}{\sqrt{\sum_{i=1}^{n} \nu_i^2 n}} \left[\sum_{i=1}^{n} \nu_i \tilde{\varepsilon}_i\right] \tag{3.35}$$

Then, because $E[\tilde{\varepsilon}_i \tilde{\varepsilon}_j] = 0$ for $i \neq j$ and $E[\tilde{\varepsilon}_i^2] = s_i^2$, the portfolio variance is

$$E\left[\left(\tilde{R}_p - E\left[\tilde{R}_p\right]\right)^2\right] = \frac{\sum_{i=1}^{n} \nu_i^2 s_i^2}{n \sum_{i=1}^{n} \nu_i^2} < \frac{\sum_{i=1}^{n} \nu_i^2 S^2}{n \sum_{i=1}^{n} \nu_i^2} = \frac{S^2}{n} \tag{3.36}$$

Thus, as n becomes large ($n \to \infty$), the variance of the portfolio goes to zero, that is, the expected return on the portfolio becomes *certain*. This implies that in the limit, the actual return equals the expected return in (3.34):

$$\lim_{n \to \infty} \tilde{R}_p = E\left[\tilde{R}_p\right] = \sqrt{\frac{1}{n} \sum_{i=1}^{n} \nu_i^2} \tag{3.37}$$

and so if there are no asymptotic arbitrage opportunities, this certain return on the portfolio must equal zero. This is equivalent to requiring

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \nu_i^2 = 0 \tag{3.38}$$

which is condition (*iii*).

We see that APT, given by the relation $a_i = \lambda_0 + \sum_{z=1}^{k} b_{iz} \lambda_z$, can be inter-

preted as a multi-beta generalization of CAPM. However, whereas CAPM says that its single beta should be the sensitivity of an asset's return to that of the market portfolio, APT gives no guidance as to what are the economy's multiple underlying risk factors. An empirical application of APT by Nai-Fu Chen, Richard Roll, and Stephen Ross (Chen, Roll, and Ross 1986) assumed that the risk factors were macroeconomic in nature, as proxied by industrial production, expected and unexpected inflation, the spread between long- and short-maturity interest rates, and the spread between high- and low-credit-quality bonds.

Other researchers have tended to select risk factors based on those that provide the "best fit" to historical asset returns.[15] The well-known Eugene Fama and Kenneth French (Fama and French 1993) model is an example of this. Its risk factors are returns on three different portfolios: a market portfolio of stocks (like CAPM), a portfolio that is long the stocks of small firms and short the stocks of large firms, and a portfolio that is long the stocks having high book-to-market ratios (value stocks) and short the stocks having low book-to-market ratios (growth stocks). The latter two portfolios capture the empirical finding that the stocks of smaller firms and those of value firms tend to have higher expected returns than would be predicted solely by the one-factor CAPM model. The Fama-French model predicts that a given stock's expected return is determined by its three betas for these three portfolios.[16] It has been criticized for lacking a theoretical foundation for its risk factors.[17]

However, there have been some attempts to provide a rationale for the Fama-

[15] Gregory Connor and Robert Korajczk (Connor and Korajczyk 1995) survey empirical tests of the APT.

[16] A popular extension of the Fama-French three-factor model is the four-factor model proposed by Mark Carhart (Carhart 1997). His model adds a proxy for stock momentum.

[17] Moreover, some researchers argue that what the model interprets as risk factors may be evidence of market inefficiency. For example, the low returns on growth stocks relative to value stocks may represent market mispricing due to investor overreaction to high growth firms. Josef Lakonishok, Andrei Shleifer, and Robert Vishny (Lakonishok, Shleifer, and Vishny 1994) find that various measures of risk cannot explain the higher average returns of value stocks relative to growth stocks.

French model's good fit of asset returns. Heaton and Lucas (Heaton and Lucas 2000) provide a rationale for the additional Fama-French risk factors. They note that many stockholders may dislike the risks of small-firm and value stocks, the latter often being stocks of firms in financial distress, and thereby requiring higher average returns. They provide empirical evidence that many stockholders are, themselves, entrepreneurs and owners of small businesses, so that their human capital is already subject to the risks of small firms with relatively high probabilities of failure. Hence, these entrepreneurs wish to avoid further exposure to these types of risks.

We will later develop another multibeta asset pricing model, namely Robert Merton's Intertemporal Capital Asset Pricing Model (ICAPM) (Merton 1973a), which is derived from an intertemporal consumer-investor optimization problem. It is a truly dynamic model that allows for changes in state variables that could influence investment opportunities. While the ICAPM is sometimes used to justify the APT, the static (single-period) APT framework may not be compatible with some of the predictions of the more dynamic (multiperiod) ICAPM. In general, the ICAPM allows for changing risk-free rates and predicts that assets' expected returns should be a function of such changing investment opportunities. The model also predicts that an asset's multiple betas are unlikely to remain constant through time, which can complicate deriving estimates of betas from historical data.[18]

3.4 Summary

In this chapter we took a first step in understanding the equilibrium determinants of individual assets' prices and returns. The Capital Asset Pricing Model

[18] Ravi Jagannathan and Zhenyu Wang (Jagannathan and Wang 1996) find that the CAPM better explains stock returns when stocks' betas are permitted to change over time and a proxy for the return on human capital is included in the market portfolio.